The Mean Value Theorem

Objectives:

- Determine if a function satisfies Rolle's Theorem.
- Determine if a function satisfies the Mean Value Theorem.

Theorem (Rolle's Theorem)**.**
Let f be a function that satisfies the following:
(1) f is continuous on the closed interval $[a, b]$.
(2) f is differentiable on the open interval (a, b).
(3) $f(a) = f(b)$.
Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.

Proof.
Case 1 $f(x) = k$ where k is a constant.
If $f(x) = k$ then $f'(x) = 0$ for every real number x. So in particular $f(x) = 0$ for all $x \in (a, b)$.

Case 2 $f(x) > f(a)$ for some $x \in (a, b)$.
Since f is continuous on $[a, b]$, f attains its maximum value by The Extreme Value Theorem. Since $f(a) = f(b)$, f must attain its maximum value at some $c \in (a, b)$. So f has a local maximum at c. Since f is differentiable on (a, b), $f'(x) = 0$ by Fermat's Theorem.

Case 3 $f(x) < f(a)$ for some $x \in (a, b)$.
Since f is continuous on $[a, b]$ f attains its minimum value by The Extreme Value Theorem. Since $f(a) = f(b)$, there exists $c \in (a, b)$ such that $f(c)$ is the absolute minimum. Since f is differentiable on (a, b), $f'(c) = 0$ by Fermat's Theorem.

So in all cases we have shown the desired result. □

Theorem (The Mean Value Theorem)**.**
Let f be a function that satisfies the following conditions:

(1) f is continuous on the closed interval $[a, b]$.
(2) f is differentiable on the open interval (a, b).

Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Equivalently $f(b) - f(a) = f'(c)(b - a)$.

Proof.
The slope of the secant line from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$m = \frac{f(b) - f(a)}{b - a}.$$

Using the point slope form the equation of the secant line may be written as

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

Solving for y we get

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Define a new function $h(x)$ where $h(x)$ is the difference between the graph of f and the graph of the secant line.

$$h(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Since $f(x)$ and the secant line are continuous on $[a, b]$, $h(x)$ is continuous on $[a, b]$, This satisfies the first hypothesis of Rolle's Theorem. Since $f(x)$ and the secant line are differentiable on (a, b), $h(x)$ is differentiable on (a, b). This satisfies the second hypothesis of Rolle's Theorem.

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - (f(b) - f(a)) = 0$$

Since $h(a) = h(b)$, the third hypothesis of Rolle's Theorem has been satisfied. So there exists $c \in (a, b)$ such that $f'(c) = 0$.

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \Rightarrow h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

Evaluating the derivative of h at c we get

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Substitute 0 for $h'(c)$ and we get

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

So

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

Example.
Verify that $f(x) = x\sqrt{x + 6}$ satisfies Rolle's Theorem on $[-6, 0]$, then find c.

Solution.
From discussion in chapter 2 we know that $g(x) =$ and $h(x) = \sqrt{x + 6}$ are continuous on their entire domain. We also know from discussion in chapter w that the product of continuous functions is continuous. So f is continuous on $[-6, 0]$ (1^{st} hypothesis).

We know that $h(x)$ is differentiable on (a, b) (2^{nd} condition of Rolle's Theorem.

$f(-6) = 0$ and $f(0) = 0$ (3^{rd} condition of Rolle's Theorem).

So Rolle's Theorem is satisfied.

$$f'(x) = x\frac{d}{dx}(x + 6)^{1/2} + (x + 6)^{1/2}\frac{d}{dx}x = \frac{1}{2}x(x + 6)^{-1/2} + (x + 6)^{1/2}$$

$$f'(c) = 0 \Rightarrow (c + 6)^{-1/2}\left(\frac{1}{2}c + (c + 6\right) = 0 \Rightarrow \frac{\frac{3}{2}c + 6}{\sqrt{c + 6}} = 0 \Rightarrow \frac{3}{2}c = -6 \Rightarrow c = -4$$

Example.
Show that $x^5 - 6x + 2 = 0$ has exactly one root in the interval $[-1, 1]$.

Solution.
$f(-1) = -1 + 6 + 2 = 7 > 0$. $f(1) = 1 - 6 + 2 = -3 < 0$. f is continuous on $[-1, 1]$. So by The Intermediate Value Theorem there exists $c \in [-1, 1]$ such that $f(c) = 0$. This shows at least one root. Now to show exactly one root we shall assume 2 roots and derive a contradiction.

Suppose a and b are both roots in $[-1, 1]$. Then $f(a) = f(b) = 0$ and f is continuous. So by Rolle's Theorem there exists $c \in [-1, 1]$ such that $f'(c) = 0$.

$$f'(x) = 5x^4 - 6$$

$$5x^4 - 6 = 0 \Rightarrow x^4 = \frac{6}{5} \Rightarrow x = \pm\sqrt[4]{\frac{6}{5}} \approx \pm 1.04$$

Since ± 1.04 is not in $[-1, 1]$, $f(x)$ can not have more than one root in $[-1, 1]$.