Limits at Infinity; Horizontal Asymptotes

Objectives:

- Evaluate the limit of a function as x gets arbitrarily large.
- Identify the horizontal asymptotes of a function.

Limits at Infinity

Example.
Suppose

$$f(x) = \frac{x^2+1}{x^2-1}$$

If we look at the graph of f we can see that for x values large in magnitude the graph gets close to the line $y = 1$. We can see this from the following table of values.

x	$y = \frac{x^2+1}{x^2-1}$
5	1.0833
10	1.0202
100	1.0002
-5	1.0833
-10	1.0202
-100	1.0002

So it appears that

$$\lim_{x\to-\infty} \frac{x^2+1}{x^2-1} = 1 \text{ and } \lim_{x\to\infty} \frac{x^2+1}{x^2-1} = 1$$

Definition.
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means the values of $f(x)$ get arbitrarily close to L by making x sufficiently large.

Definition.
Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means the values of $f(x)$ get arbitrarily close to L by making x sufficiently large in the negative direction.

Horizontal Asymptotes

Definition.
The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

Example.
Let

$$f(x) = \frac{1}{x}$$

If we look at the graph of f we can see that $y = 0$ is the horizontal asymptote to the curve. This is further evidenced by the following table of values.

x	$y = \frac{1}{x}$
10	0.1
100	0.01
1000	0.001
-10	-0.1
-100	-0.01
-1000	-0.001

Theorem. If $r > 0$ is any rational number. then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number and x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example.
Find the limit if it exists.

$$\lim_{x\to\infty} \frac{6x^2 + 5x}{-2x^2 + 5x - 3}$$

Solution. We want to take advantage of the previous theorem. To do this we shall multiply each term in the numerator and the denominator by the reciprocal of the highest power of x.

$$\lim_{x\to\infty} \frac{6x^2 + 5x}{-2x^2 + 5x - 3} =$$

$$\lim_{x\to\infty} \frac{6x^2 + 5x}{-2x^2 + 5x - 3} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} =$$

$$\lim_{x\to\infty} \frac{6 + \frac{5}{x}}{-2 + \frac{5}{x} - \frac{3}{x^2}} =$$

$$\frac{6 + 0}{-2 + 0 - 0} = -3$$