The Fundamental Theorem of Calculus

Objectives:

- Use the fundamental theorem to evaluate definite integrals.

Consider the following definite integral.

$$\int_a^b xdx$$

If we evaluate this integral using right endpoints we get

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i^* = \frac{i(b-a)}{n}$$

Hence

$$\int_a^b xdx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{i(b-a)}{n}\right)\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \sum_{i=1}^{n} \frac{i(b-a)}{n}\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \frac{b-a}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{b-a}{n}\left(an + \frac{b-a}{n}\cdot\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{n}\cdot\frac{n+1}{2}\right]$$

$$= \lim_{n\to\infty} \left[ab - a^2 + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{2n}\right]$$

$$= ab - b^2 + \frac{(b-a)^2}{2}$$

$$= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2}$$

$$= \frac{b^2 - a^2}{2}$$

Notice that if $f(x) = x$ then one antiderivative is $F(x) = \frac{1}{2}x^2$. Notice that $F(b) - F(a) = \frac{b^2}{2} - \frac{a^2}{2}$ which is what we got taking the limit of the Riemann sums. Do you suppose it always works this way? It appears there is a connection between integration and differentiation.

Theorem (The Fundemantatl Theorem of Calculus)**.**
Suppose $f(x)$ is continuous on the interval $[a, b]$.
1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$.

Example.
Find the derivative of the following function using part 1 of the Fundamental Theorem. Then find the derivative by using part 2 of the Fundamental Theorem.

$$g(x) = \int_\pi^x (2 + \cos t)dt$$

Solution.
By part 1 of the Fundamental Theorem

$$g'(x) = \frac{d}{dx}\left[\int_\pi^x (2 + \cos t)dt\right] = 2 + \cos x$$

By part 2 of the Fundamental Theorem

$$g(x) = \int_\pi^x (2 + \cos t)dt = 2t + \sin t]_\pi^x = (2x - \sin x) - (2\pi + \sin \pi) = 2x + \sin x - 2\pi$$

$$g'(x) = \frac{d}{dx}[2x + \sin x - 2\pi] = 2 + \cos x$$

Example.
Use part 1 of the Fundamental Theorem to find the derivative.

$$h(x) = \int_0^{x^2} \sqrt{1 + r^3}dr$$

Solution. Since the upper limit of integration is a function of x we must be aware of the chain rule when taking the derivative. Let $u = x^2$. Then $\frac{du}{dx} = 2x$. From the chain rule $\frac{dh}{dx} = \frac{dh}{du}\frac{du}{dx}$. Hence

$$\frac{d}{dx}\left[\int_0^{x^2} \sqrt{1 + r^2}dr\right] = \frac{d}{du}\left[\int_0^u \sqrt{1 + r^3}dr\right] \cdot \frac{du}{dx} = \sqrt{1 + u^3} + \frac{du}{dx} = 2x\sqrt{1 + x^6}$$

Example.
Evaluate the definite integral.

$$\int_{-1}^{3} x^5 dx$$

Solution.
First we find the antiderivative of x^5, then we evaluate F at the two limits of integration and take their difference.

$$\int_{-1}^{3} x^5 dx = \frac{1}{6}x^6\Big]_{-1}^{3} = \frac{1}{6}(3)^3 - \frac{1}{6}(-1)^3 = \frac{364}{3}$$

Example.
Evaluate the definite integral.

$$\int_{\pi}^{2\pi} \cos\theta d\theta$$

Solution.

$$\int_{\pi}^{2\pi} \cos\theta d\theta = \sin\theta\Big]_{\pi}^{2\pi} = 0 - 0 = 0$$