10.3 Separable Equations

Objectives:

- Solve separable differential equations.
- Solve equations given initial conditions.

An equation containing first order derivatives is called a **first order differential equation**. A **separable equation** is a first order equation in which the expression for dy/dx can be factored into a function of x times a function of y.

$$\frac{dy}{dx} = g(x)f(y)$$

We call this equation separable because we can separate the differentials. Let $f(y) = 1/h(y)$. Then the equation becomes

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

Now we can separate the differential to get

$$h(y)\,dx = g(x)\,dx$$

We solve the equation by integrating both sides.

Example.
Solve the following differential equation.

$$\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}}$$

Solution.
First we separate the differentials to get

$$y\sqrt{1+y^2}\,dy = te^t\,dt$$

Now we integrate both sides to get

$$\int y\sqrt{1+y^2}\,dy = \int te^t\,dt$$

The integral on the left requires a u substitution with $u = 1+y^2$ and $\frac{1}{2}du = y\,dy$. The integral on the right requires integration by parts with $u = t$ and $dv = e^t\,dt$.

$$y\sqrt{1+y^2}\,dy = te^t\,dt \Rightarrow$$

$$\frac{1}{2}\cdot\frac{2}{3}\left(1+y^2\right)^{3/2} = te^t - \int e^t\,dt \Rightarrow$$

$$\frac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$$

The problem gave us the rate of change in y with respect to t. When we separate the variables and integrate we get an equation that involves x and t. Sometimes this is the best we can do, but we should write y as an explicite rule of t if possible. We need only one constant of integration since a constant on each side can be combined into one.

$$\frac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C \;\Rightarrow$$
$$\left(1+y^2\right)^{3/2} = 3\left(te^t - e^t + C\right) \;\Rightarrow$$
$$1+y^2 = \left[3\left(te^t - e^t + C\right)\right]^{2/3} \;\Rightarrow$$
$$y^2 = \left[3\left(te^t - e^t + C\right)\right]^{2/3} - 1 \;\Rightarrow$$
$$y = \pm\sqrt{\left[3\left(te^t - e^t + C\right)\right]^{2/3} - 1}$$

Example.
Find the solution of the differential equation that satisfies the given initial condition.

$$x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} = 0, \qquad y(0) = 1$$

Solution.

$$x + 2y\sqrt{x^2+1}\,\frac{dy}{dx} = 0 \;\Rightarrow$$
$$2y\sqrt{x^2+1}\,\frac{dy}{dx} = -x \;\Rightarrow$$
$$2y\,dy = -\frac{x}{\sqrt{x^2+1}}\,dx \;\Rightarrow$$
$$\int 2y\,dy = -\int \frac{x}{\sqrt{x^2+1}}\,dx$$

On the right side let $u = x^2+1$ so $\frac{1}{2}\,du = x\,dx$

$$y^2 = -\frac{1}{2}(2)\left(x^2+1\right)^{1/2} + C \;\Rightarrow$$
$$y^2 = -\sqrt{x^2+1} + C$$

Now we use the initial condition to find C.

$$y(0) = 1 \;\Rightarrow\; 1^2 = -\sqrt{0^2+1} + C \;\Rightarrow\; C = 2$$
$$y^2 = -\sqrt{x^2+1} + 2$$