12.6 Absolute Convergence and the Ratio and Root Tests

Objectives:

- Determine if a convergent series is absolutely convergent or conditionally convergent.
- Use the ratio test to determine if a series is convergent or divergent.
- Use the root test to determine if a series is convergent or divergent.

Absolute Convergence

Given any series $\sum a_n$ there is a corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| = \cdots$$

whose terms are the absolute values of the terms in the original series.

Definition.
A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Example.
The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

is absolutely convergent since the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

is convergent.

Example.
The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} - \frac{1}{4} + \cdots$$

is not absolutely convergent since the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Definition.
A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem.
If a series is absolutely convergent then it is convergent.

Example.
Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

Solution.
The series is conditionally convergent by the alternating series test and the p series test since

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \text{diverges}$$

The Ratio Test

Theorem (The Ratio Test)**.**

$$(i) \text{ If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent}$$

$$(ii) \text{ If } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \text{ or } \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \text{ then } \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

Proof.
Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$. Then there exists a constant r such that $L < r < 1$. Likewise there exist a positive integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| < r \qquad \text{whenever} \qquad n \geq N$$

Multiply both sides of the inequality by $|a_n|$ to get $|a_{n+1}| < r|a_n|$ whenever $n \geq N$. Let $n = N$ to get $|a_{N+1}| < |a_N|r$. Let $n = N+1$ to get $|a_{N+2}| < |a_{N+1}|r$. Since $|a_{N+1}| < |a_N|r$ and $r > 0$, $|a_{N+1}|r < |a_N|r^2$. So When $n = N+2$ we get

$$|a_{N+2}| < |a_{N+1}|r < |a_N|r^2$$

Continuing in this manner for k more iterations we can generalize the following inequality.

$$|a_{N+k}| < |a_N|r^k$$

Since $0 < r < 1$ and $|a_N|$ is fixed, the geometric series

$$\sum_{k=1}^{\infty} |a_N|r^k = |a_n|r + |a_n|r^2 + |a_n|r^3 + \cdots$$

converges. Since $|a_{N+1}| < |a_N|r$, by the comparison test the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is convergent.

This allows us to rewrite the series as follows:

$$\sum_{n-1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{k=1}^{\infty} |a_{N+k}|$$

The first sum has a finite number of terms and the second sum is convergent so

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent. This means $\sum a_n$ is absolutely convergent.

For part two see the book.

□

Example.
Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} e^{-n} n!$$

Solution.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!/e^{n+1}}{n!/e^n} = \frac{e^n(n+1)!}{e^{n+1}n!} = \frac{n+1}{e}$$

Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{n+1}{e} = \infty$$

So the series is divergent by the ratio test.

The Root Test

Theorem (The Root Test)**.**

$$(i) \text{ If } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1, \text{ then } \sum_{n=1}^{\infty} a_n \text{ is absolutely convergent.}$$

$$(ii) \text{ If } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1, \text{ then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Proof.
(i) Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ there exists a positive constant r such that $L < r < 1$. There exists a positive integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geq n$. Since $\sqrt[n]{|a_n|} < r$, $|a_n| < r^n$. Since $r < 1$ the geometric series $\sum r^n$ converges. By the comparison test the series $\sum |a_n|$. Hence the series $\sum a_n$ is absolutely convergent. □

Example.
Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$$

Solution.
This series is obviously convergent from the alternating series test, but we need an example using the root test, so here it is.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$$

So the series converges by the root test.